Problem 1. Calculate the norms of the following operators:
(a) $f \in L^\infty[0,1]$, $M_f : L^2[0,1] \to L^2[0,1]$, $(M_f g)(x) = f(x)g(x)$.
(b) $g \in C[0,1]$, $T_g : C[0,1] \to \mathbb{C}$, $T_g x = \int_0^1 g(t)x(t)\,dt$.

Problem 2. Show that the convolution $f * g(x) = \int f(x-y)\,g(y)\,dy$ of two functions $f, g \in L_1(\mathbb{R})$ is well-defined for $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$.

Problem 3. Show that every bounded sequence in a Hilbert space has a weakly convergent subsequence.

Problem 4. A linear operator $T \in B(\mathcal{H})$ *achieves its norm* if there is a unit vector $x \in \mathcal{H}$ such that $\|T\| = \|Tx\|$. Give examples of bounded self-adjoint operators that do *not* achieve their norm and
(a) have an orthonormal basis of eigenvectors,
(b) does not have eigenvectors.

Problem 5. Let $\mathcal{H}$ be a Hilbert space and $K \subseteq \mathcal{H}$ be a convex subset. An element $x \in K$ is called an *extreme point* if for all $x_0, x_1 \in K$ and for all $\lambda \in]0,1[$

$$(1-\lambda)x_0 + \lambda x_1 = x \implies x_0 = x_1 = x.$$

(a) Show that every unit vector is an extreme point of the unit sphere $B_{\mathcal{H}}(0,1)$.
(b) Show that every isometric linear operator $V \in B(\mathcal{H})$ is an extreme point of the unit sphere $B_{B(\mathcal{H})}(0,1)$.

Problem 6. Let $\mathcal{H}$ be a Hilbert space with scalar product $\langle .,. \rangle$ and $A \in B(\mathcal{H})_+$ a positive semi-definite linear operator. Show the following:
(a) that

$$\langle x, y \rangle_A := \langle Ax, y \rangle$$

defines a positive semi-definite sesquilinear form,
(b)

$$\|A\| = \min\{a \in \mathbb{R} \mid A \leq aI\},$$

(c) the inner product $\langle .,. \rangle_A$ is positive definite if and only if $\ker A = \{0\}$,
(d) in the case where $\ker A = \{0\}$ holds, the induced norm $\|x\|_A = \langle Ax, x \rangle^{1/2}$ is equivalent to the original norm $\|x\| = \langle x, x \rangle^{1/2}$ if and only if A has a bounded inverse,
(e) Let A be invertible and $T \in B(\mathcal{H})$ be another operator with adjoint T^*. Find a formula for the adjoint of T with respect to the scalar product $\langle .,. \rangle_A$.